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On Chained Relations

Chained Relations

Relations are comparison operators, such as =, <, >, ≤, ≥, ≠, etc. They are equivalent to verbs in standard English ("is equal to", "is lesser than", etc.). For a fully-formed mathematical statement, a relation is required; just as a verb is required to avoid a sentence fragment in English.

It's common to write compound statements as chained relations. For example, $a = b = c$ means that $a = b$ and $b = c$; note the conjunction "and", which is always implied by a chained relation. By the transitive property of equality, this implies that $a = c$. If a statement becomes very long, then we may break it up across multiple lines (exactly as is done in a programming language like C++). Customarily the breaks start with the next relation symbol. Make sure you can read the broken-up chain as a single (compound) statement. For example, you may see this:

$$\begin{aligned} -(ab) &= (-1)(ab) \\ &= (-1a)(b) \\ &= (-a)(b) \end{aligned}$$

This means: $-(ab) = (-1)(ab)$, and $(-1)(ab) = (-1a)(b)$, and $(-1a)(b) = (-a)(b)$. We might succinctly read this as: "$-(ab)$ is equal to $(-1)(ab)$, which is equal to $(-1a)(b)$, which is equal to $(-a)(b)$". By the transitivity and symmetry properties, this implies that $(-a)(b) = -(ab)$.

It is also common to give explanations for the transformation at each step in sidebar comments to the right of each line; note that these are *not* part of the algebraic statement (nor do they break it up); there is still just one single compound statement being shown. For example:

$$\begin{aligned} -(a+b) &= (-1)(a+b) && \text{by Theorem 2} \\ &= (-1a)+(-1b) && \text{by the distributive property} \\ &= (-a)+(-b) && \text{by Theorem 2} \\ &= -a-b && \text{by the definition of subtraction} \end{aligned}$$

We can also chain together other relational statements. For example, $a < b < c$ means $a < b$ and $b < c$, which by transitivity of the lesser-than relation implies that $a < c$. Statements like these are only meaningful if all of the inequality operators point in a single direction. For example, (1) $a < b < c$, (2) $a \le b \le c$, (3) $a < b \le c$, (4), $a \le b < c$, (5) $a > b > c$, (6) $a \ge b \ge c$, (7) $a > b \ge c$, and (8) $a \ge b > c$, are all acceptable. We should not write a statement like $a > b < c$, because transitivity does not apply, and so no conclusion about the relation of a and c can be made.

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Common Errors

Consider the following short theorem and proof.

Theorem: $(a+b)^2=a^2+2ab+b^2$

Proof:

$(a+b)^2=(a+b)(a+b)$	definition of exponent
$=a(a+b)+b(a+b)$	by the distributive property
$=a^2+ab+ab+b^2$	by the distributive property
$=a^2+2ab+b^2$	combine like terms

Common Error #1

It's an error to omit the connecting relation between lines. In this case, no clear relational statement is being made (all we have are a series of expression fragments).

$$(a+b)^2$$
$$(a+b)(a+b)$$
$$a(a+b)+b(a+b)$$
$$a^2+ab+ab+b^2$$
$$a^2+2ab+b^2$$

Connecting relations are missing

Common Error #2

It's an error to collapse the theorem statement and proof together. This has the appearance of assuming that the goal to be proved is true at the start of the proof. Note that the 2nd expression appears twice in the sequence ("circular reasoning"), and no justification for the 2nd step exists.

$$(a+b)^2=a^2+2ab+b^2$$
$$=(a+b)(a+b)$$
$$=a(a+b)+b(a+b)$$
$$=a^2+ab+ab+b^2$$
$$=a^2+2ab+b^2$$

**Collapsed theorem and proof
(circular reasoning?)**

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Common Error #3

It's an error to write the proof as a series of separate equations, instead of a single chained relation. Like error #2, this starts off with the appearance of assuming the thing we're trying to prove – consider that the reader can't verify any of the statements as being true until the *end* of sequence. Furthermore, there's a lot of unneccessary writing as compared to the chained-relation format. Keep in mind that we're not solving equations here.

$$(a+b)^2=a^2+2ab+b^2$$
$$(a+b)(a+b)=a^2+2ab+b^2$$
$$a(a+b)+b(a+b)=a^2+2ab+b^2$$
$$a^2+ab+ab+b^2=a^2+2ab+b^2$$
$$a^2+2ab+b^2=a^2+2ab+b^2$$

Separate equations, not a chained relation (circular reasoning?)

Online Practice Quiz

Consider practicing reading chained relations with the short online quiz linked below. Retry it as many times as needed until you can reliably get the all the answers correct, in the indicated time, on every attempt.

http://www.automatic-algebra.org/chainedrelations.htm