Daniel R. Collins – dcollins@superdan.net

WHY Z-SCORES HAVE MEAN 0 AND STANDARD DEVIATION 1

Numerical Example

We'll start with a short numerical example. Pick any two numbers for a population; I'll choose x = {2, 6}. Compute the mean and stdev of this population:

$$\mu = \frac{\sum x}{N} = \frac{8}{2} = 4$$

$$\sigma = \sqrt{\frac{\sum (x-\mu)^2}{N}} = \sqrt{\frac{(-2)^2 + (2)^2}{2}} = \sqrt{\frac{8}{2}} = \sqrt{4} = 2$$

Convert all of the population to z-scores:

$$z_1 = \frac{x-\mu}{\sigma} = \frac{2-4}{2} = -2/2 = -1$$

$$z_2 = \frac{x-\mu}{\sigma} = \frac{6-4}{2} = 2/2 = 1$$

Now find the mean and stdev of this set of z-scores {−1, 1}:

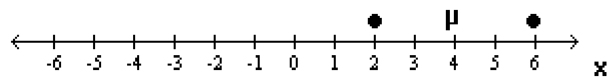
$$\mu_z = \frac{\sum z}{N} = \frac{-1+1}{2} = \frac{0}{2} = 0$$

$$\sigma_z = \sqrt{\frac{\sum (z-\mu_z)^2}{N}} = \sqrt{\frac{(-1)^2 + (1)^2}{2}} = \sqrt{\frac{2}{2}} = \sqrt{1} = 1$$

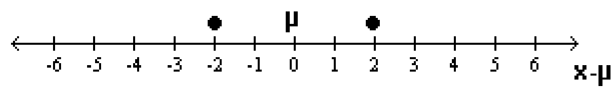
So we see that, as expected, the mean and stdev of the z-scores are 0 and 1.

Graphical Illustration

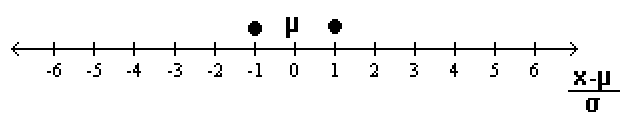
Many people find the graphical illustration to be the most helpful. I'll look at the same population as before: x = {2, 6}, with its mean of μ = 4:



The first operation in the z-score process is to subtract μ; in this case it reduces all of the values by 4. Of course, this also reduces the center by 4, which indicates that the new mean must be at 0. This would be true for any population, since taking any μ−μ = 0. (In a precalculus or trigonometry class this subtraction is called a "horizontal shift" of the graph.)



Now notice that the standard deviation of the points from the center is 2; in this simple example, both of the points are in fact exactly 2 units from the center. So if we divide by that number, then the distance from the center becomes 1; in this case, 2/2=1, but likewise for any population, σ/σ = 1. (In a precalculus or trigonometry class this division would be called a "horizontal compression" of the graph.)



So you can see that after these two operations, the points have been shifted to be centered at 0 (because μ−μ = 0), and have a standard distance from the center of 1 (because σ/σ = 1).

Algebraic Proof

Of course, the preceding were just examples, which don't exactly prove that standardizing works like this in all cases (although hopefully the graphical illustration gives some intuition that it must be the case). Here is a formal proof that standardizing shifts the mean to zero in all cases.

First of all, note that the summation operator distributes over terms just like multiplication (they're both repeated additions, so they work the same in this sense). By this I mean ∑(a+b) = ∑a + ∑b and so forth.

Now, consider any population of x's, with the mean and z-scores defined as usual: μ = ∑x/N and z = (x-μ)/σ. Let's think about computing the mean of the z-scores, μ_z:

$$\mu_z = \frac{\sum z}{N} = \frac{\sum (x-\mu)/\sigma}{N}$$

But the factor ∑(x – μ), the sum of all the differences from the mean, is itself always zero. To see this, consider the value of ∑μ; since that's just adding up N copies of μ, it's the same as ∑x:

$$\sum \mu = N \cdot \mu = N \cdot \frac{\sum x}{N} = \sum x$$

And therefore the sum of differences ∑(x – μ) = ∑x – ∑μ = ∑x – ∑x = 0. So this allows us to complete the proof that the mean of the z-scores is always zero:

$$\mu_z = \frac{\sum z}{N} = \frac{\sum (x-\mu)/\sigma}{N} = \frac{0/\sigma}{N} = 0$$

More Resources

Hopefully, that clarifies why the z-score formula works the way it does. If you like, here are some other places that you can consult:

- The Weiss *Introductory Statistics* textbook, Section 3.4, has a similar numerical example on standardizing a population of data points (using 5 points instead of just 2).
- The following is an interesting worksheet with some more examples (looking at the top “Exploration IV”): http://faculty.frostburg.edu/math/monline/stat/33_p2.html
- I left out the algebraic proof that the stdev of z-scores is always 1 (σ_z = 1). It's a bit more technical, but can be done.
- Alternatively, you can look at the end of this short 4-page article on z-scores, where the proof is written very concisely (but using different symbols, x = Y, μ = M, σ = S):

  http://www.utdallas.edu/~herve/Abdi-Zscore2007-pretty.pdf